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Quantum Yang-Baxter equation and constant *R*-matrix over Grassmann algebra

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Abstract: Constant solutions to Yang-Baxter equation are investigated over Grassmann algebra for the case of 6-vertex ***R***-matrix. The general classification of all possible solutions over Grassmann algebra and particular cases with 2,3,4 generators are studied. As distinct from the standard case, when ***R***-matrix over number field can have a maximum 5 nonvanishing elements, we obtain over Grassmann algebra a set of new full 6-vertex solutions. The solutions leading to regular ***R***-matrices which appear in weak Hopf algebras are considered.

Key words: Constant solution, Grassmann algebra, Regularity, ***R***-matrix
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INTRODUCTION

The Yang-Baxter equation (Yang, 1967; Baxter, 1972) is one of the most valuable concepts in modern theoretical physics (Baxter, 1982). The importance of detailed study of Yang-Baxter equation solutions is due to its key role in constant solution models of statistics mechanics (Baxter, 1972; 1982) and field theory in lower dimensions (Yang, 1967), conformal field theory (Di Francesco *et al*., 1997) and in quantum integrable systems (Faddeev *et al*., 1990). From the group-theoretical viewpoint, when classical Yang-Baxter equation is closely connected with classical (semi-simple) group theory, the quantum Yang-Baxter equation is the basis of modern quantum group theory (Drinfeld, 1987; Shnider and Sternberg, 1993; Demidov, 1998; Chari and Pressley, 1996). There exist a constant, one-parameter and double-parameter forms of the quantum Yang-Baxter equation (Lambe and Radford, 1997). The corresponding constant (and commutative) solutions to the Yang-Baxter equation (Hietarinta, 1992) are used in evolution of non-linear equation quantization, quantum group theory (Etingof *et al*., 1997; 1999; Lu *et al*., 2000; Gu, 1997) and knot theory (Kauffman, 1991; Turaev, 1994). The solution of the quantum Yang-Baxter equation is ***R***-matrix (Kassel, 1995; Majid, 1995) [corresponding to the transfer-matrix in the lattice statistical models (Baxter, 1982)].

Recently the unitary Yang-Baxter equation solutions found an application in quantum calculations (Holievo, 2002; Kitajev *et al*., 1999), while unitary ***R***-matrix of a special type, acting on quantum state of two qubits, according to Brylinskis' theorem (Brylinski and Brylinski, 1994) can be treated as the universal quantum gate (Kauffman and Lomonaco, 2004; Zhang *et al*., 2005; Dye, 2003; Duplij *et al*., 2005).

The generalization of inverse quantum scattering method on supersymmetric systems (Khoroshkin and Tolstoy, 1991) and corresponding ***R***-matrices were considered in (Chang *et al*., 1992; Zhang and Gould, 1991). The building of supersymmetric analogues of the given constructions demands thorough consideration of Yang-Baxter equation solutions over the Grassmann algebra.

The constant Yang-Baxter equation solutions for the case of 6 vertices used for the description of the two-parameter quantum plane (Manin, 1989) and the quantum gates of special type were considered in this work (Chang *et al*., 1992; Zhang and Gould, 1991; Manin, 1989). Classification of general solutions is made and some particular cases are considered. In contrast to the standard case, when the **R**-matrix over the standard number field (for example, $\mathbb{R}$, $\mathbb{C}$) can have not more than 5 non-zero elements (Hietarinta, 1992; 1993) (actually 5-vertex solution), in our case (over the Grassmann algebra), all 6 elements can be non-zero. Thus, a new type of solution appears, which is absent in standard case (Hietarinta, 1992; 1993), is really complete 6-vertex solution. In the conclusion, the solutions leading to regular **R**-matrices appearing in weak Hopf algebras (Li and Duplij, 2002; Duplij and Li, 2001a) are considered.

YANG-BAXTER EQUATION OVER GRASSMANN ALGEBRA

Let $\boldsymbol{V}$ be a vector space, then on the tensor product $\boldsymbol{V}^{\otimes n}$ we define a linear operator $\boldsymbol{R}:\boldsymbol{V}^{\otimes n}\rightarrow\boldsymbol{V}^{\otimes n}$ in the following way. Let $\{\boldsymbol{e}_i\}$ be a basis in $\boldsymbol{V}$. Then we assign to the operator $\boldsymbol{R}$ a numerical matrix $\boldsymbol{R}$ with n index pairs

$$\boldsymbol{R}(\boldsymbol{e}_{i_1}\otimes\cdots\otimes\boldsymbol{e}_{i_n})=\boldsymbol{R}_{i_1\cdots i_n}^{j_1\cdots j_n}(\boldsymbol{e}_{j_1}\otimes\cdots\otimes\boldsymbol{e}_{j_n}), \tag{1}$$

where over repeated indices summation is supposed. Consider n-simplex equation over the tensor product $\boldsymbol{V}^{[\otimes n(n+1)/2]}$, where the linear operators $\boldsymbol{R}$ act trivially, for example, $\boldsymbol{R}_{12}(\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\boldsymbol{e}_{i_3})=r_{i_1i_2}^{j_1j_2}(\boldsymbol{e}_{i_1}\otimes\boldsymbol{e}_{i_2}\otimes\boldsymbol{e}_{i_3})$. In the general case with $K_\alpha\in\{1,\cdots,N\}$, $N=n(n+1)/2$, the $\boldsymbol{R}$ operators are:

$$(\boldsymbol{R}_{K_1\ldots K_n})_{i_1\cdots i_N}^{j_1\cdots j_N}=r_{i_{K_1}\cdots i_{K_n}}^{j_{K_1}\cdots j_{K_n}}\prod_{k=1,\ k\neq K_\alpha,\forall\alpha}^{N}\delta_{i_k}^{j_k}, \tag{2}$$

where $r_{i_{K_1}\cdots i_{K_n}}^{j_{K_1}\cdots j_{K_n}}$ is the matrix element of $\boldsymbol{R}$-matrix. For example, 2-simplex constant equation is defined by Eq.(3)

$$\boldsymbol{R}_{12}\boldsymbol{R}_{13}\boldsymbol{R}_{23}=\boldsymbol{R}_{23}\boldsymbol{R}_{13}\boldsymbol{R}_{12}, \tag{3}$$

and is called Yang-Baxter equation (Baxter, 1982; Lambe and Radford, 1997), and 3-simplex equation below

$$\boldsymbol{R}_{123}\boldsymbol{R}_{145}\boldsymbol{R}_{246}\boldsymbol{R}_{356}=\boldsymbol{R}_{356}\boldsymbol{R}_{246}\boldsymbol{R}_{145}\boldsymbol{R}_{123}, \tag{4}$$

is called tetrahedron equation in (Hietarinta, 1997) and 4-simplex equations was considered

$$\boldsymbol{R}_{1234}\boldsymbol{R}_{1567}\boldsymbol{R}_{2589}\boldsymbol{R}_{3680}\boldsymbol{R}_{4790}=\boldsymbol{R}_{4790}\boldsymbol{R}_{3680}\boldsymbol{R}_{2589}\boldsymbol{R}_{1567}\boldsymbol{R}_{1234}. \tag{5}$$

In terms of multiindex matrices, defined in Eq.(2), operator Eqs.(3)~(5) are as follows:

$$r_{j_2j_3}^{k_2k_3}\ r_{j_1k_3}^{k_1l_3}\ r_{k_1k_2}^{l_1l_2}=r_{j_1j_2}^{k_1k_2}\ r_{k_1j_3}^{l_1k_3}\ r_{k_2k_3}^{l_2l_3}, \tag{6.1}$$

$$r_{j_3j_5j_6}^{k_3k_5k_6}\ r_{j_2j_4k_6}^{k_2k_4l_6}\ r_{j_1k_4k_5}^{k_1l_4l_5}\ r_{k_1k_2k_3}^{l_1l_2l_3}=r_{j_1j_2j_3}^{k_1k_2k_3}\ r_{k_1j_4j_5}^{l_1k_4k_5}\ r_{k_2k_4j_6}^{l_2l_4k_6}\ r_{k_3k_5k_6}^{l_3l_5l_6}, \tag{6.2}$$

$$\begin{aligned}&r_{j_4j_7j_9j_0}^{k_4k_7k_9k_0}\ r_{j_3j_6j_8k_0}^{k_3k_6k_8l_0}\ r_{j_2j_5k_8k_9}^{k_2k_5l_8l_9}\ r_{j_1k_5k_6k_7}^{k_1l_5l_6l_7}\ r_{k_1k_2k_3k_4}^{l_1l_2l_3l_4}=\\&r_{j_1j_2j_3j_4}^{k_1k_2k_3k_4}\ r_{k_1j_5j_6j_7}^{l_1k_5k_6k_7}\ r_{k_2k_5j_8j_9}^{l_2l_5k_8k_9}\ r_{k_3k_6k_8j_0}^{l_3l_6l_8k_0}\ r_{k_4k_7k_9k_0}^{l_4l_7l_9l_0}.\end{aligned} \tag{6.3}$$

General formulation of similar (n-simplex) equations was given in (Carter and Saito, 1996), and permutation equations were studied in (Hietarinta, 1993; 1997).

Here we will consider constant solutions for 2-simplex Eq.(6.1) (Yang-Baxter equation) over Grassmann algebra, being a particular case of a superalgebra (Berezin, 1983; Kac, 1977), which is an important step on the way of consistent supersymmetric generalization of Yang-Baxter equation and its constant solutions (Khoroshkin and Tolstoy, 1991; Zhang, 1991; Links *et al*., 1994).

Let Λ be a commutative superalgebra over a field K (where K=$\mathbb{R}$, $\mathbb{C}$) with the dissociation on the direct sum $\Lambda=\Lambda_{\bar{0}}\oplus\Lambda_{\bar{1}}$ (Berezin, 1983; Kac, 1977). Elements a from $\Lambda_{\bar{0}}$ and $\Lambda_{\bar{1}}$ are homogeneous with regard to parity $p(a)\overset{\text{def}}{=}\{\bar{i}\in\{\bar{0},\ \bar{1}\}=Z_2\mid a\in\Lambda_{\bar{i}}\}$. Supercommutator is defined as $[a,b]=ab-(-1)^{p(a)p(b)}ba$. In the particular case Λ_n is the Grassmann algebra with the element $\xi_1,\cdots,\xi_n$, which satisfies $\xi_i\xi_j+\xi_j\xi_i=0$, $1\leq i, j\leq n$, in particular $\xi_i^2=0$ (n can be infinite). Superalgebra structure in Λ_n is defined by the fact that parity of the element is considered equal to $p(\xi_i)=\bar{1}$ (Leites, 1980). Then even $x\in\Lambda_{\bar{0}}$ and odd

$\chi \in \Lambda_{\bar{1}}$ elements of Grassmann algebra can be expanded into the series (which is finite under finite number of elements ξ_i)

$$x = x_{\text{numb}} + x_{\text{nil}} = x_0 + x_{12}\xi_1\xi_2 + x_{13}\xi_1\xi_3 + \cdots = x_{\text{numb}} + x_{i_1\ldots i_{2r}}\xi_{i_1}\cdots\xi_{i_{2r}}, \tag{7.1}$$

$$\chi = \chi_{\text{nil}} + x_1\xi_1 + x_2\xi_2 + \cdots + x_{123}\xi_1\xi_2\xi_3 + \cdots = x_{i_1\ldots i_{2r-1}}\xi_{i_1}\cdots\xi_{i_{2r-1}}, \tag{7.2}$$

where $x_{i_1\ldots i_n} \in K$. The mapping ε, discarding odd elements, is called numerical mapping (Manin, 1984; De Witt, 1992) [canonical projection (Leites, 1983), body map (Rogers, 1980; Rabin, 1987; 1991)] and it acts on the elements in Eq.(7) as $\varepsilon(x) = x|_{\xi_i=0} = x_{\text{numb}}$, $\varepsilon(\chi) = \chi|_{\xi_i=0} = 0$. From Eq.(7) it follows that, for example, the equations $x^2=0$, $\chi x=0$ and $\chi\chi'=0$ can have non-zero and non-trivial solutions (zero divisors and nilpotents), which can essentially extend the number of solutions for different equations including the Yang-Baxter equation. For example, in Λ_4 even non-zero nilpotents $x^2=0$ satisfy

$$x_0 = 0,\ x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0, \tag{8}$$

and for component non-zero zero dividers $\chi x=0$, we get

$$\left.\begin{aligned} x_0 &= 0,\\ x_1x_{23} - x_2x_{13} + x_3x_{12} &= 0,\\ x_1x_{24} - x_2x_{14} + x_4x_{12} &= 0,\\ x_1x_{34} - x_3x_{14} + x_4x_{13} &= 0,\\ x_2x_{34} - x_3x_{24} + x_4x_{23} &= 0. \end{aligned}\right\} \tag{9}$$

For $\chi\chi'=0$ we get the conditions

$$x_ix'_j - x_jx'_i = 0,\ \ i,j = 1,2,3,4 \tag{10}$$

which shows that such odd objects in Eq.(7.2) are nilpotents of the second nilpotency degree $n^2=0$.

Let us consider $\boldsymbol{R}$-matrix over even part of Grassmann algebra, for example, with 4 elements, let us write its dissociation onto numerical and nilpotent parts

$$\boldsymbol{R} = \boldsymbol{R}^{(0)} + \boldsymbol{R}^{(12)}\xi_1\xi_2 + \boldsymbol{R}^{(13)}\xi_1\xi_3 + \boldsymbol{R}^{(14)}\xi_1\xi_4 + \boldsymbol{R}^{(23)}\xi_2\xi_3 + \boldsymbol{R}^{(24)}\xi_2\xi_4 + \boldsymbol{R}^{(34)}\xi_3\xi_4 + \boldsymbol{R}^{(1234)}\xi_1\xi_2\xi_3\xi_4, \tag{11}$$

the components of Yang-Baxter equation can be represented in the following way

$$\left.\begin{aligned} \boldsymbol{R}_{12} &= \boldsymbol{R}_{12}^{(0)} + \boldsymbol{R}_{12}^{(12)}\xi_1\xi_2 + \boldsymbol{R}_{12}^{(13)}\xi_1\xi_3 + \boldsymbol{R}_{12}^{(14)}\xi_1\xi_4 + \\ &\quad \boldsymbol{R}_{12}^{(23)}\xi_2\xi_3 + \boldsymbol{R}_{12}^{(24)}\xi_2\xi_4 + \boldsymbol{R}_{12}^{(34)}\xi_3\xi_4 + \boldsymbol{R}_{12}^{(1234)}\xi_1\xi_2\xi_3\xi_4,\\ \boldsymbol{R}_{13} &= \boldsymbol{R}_{13}^{(0)} + \boldsymbol{R}_{13}^{(12)}\xi_1\xi_2 + \boldsymbol{R}_{13}^{(13)}\xi_1\xi_3 + \boldsymbol{R}_{13}^{(14)}\xi_1\xi_4 + \\ &\quad \boldsymbol{R}_{13}^{(23)}\xi_2\xi_3 + \boldsymbol{R}_{13}^{(24)}\xi_2\xi_4 + \boldsymbol{R}_{13}^{(34)}\xi_3\xi_4 + \boldsymbol{R}_{13}^{(1234)}\xi_1\xi_2\xi_3\xi_4,\\ \boldsymbol{R}_{23} &= \boldsymbol{R}_{23}^{(0)} + \boldsymbol{R}_{23}^{(12)}\xi_1\xi_2 + \boldsymbol{R}_{23}^{(13)}\xi_1\xi_3 + \boldsymbol{R}_{23}^{(14)}\xi_1\xi_4 + \\ &\quad \boldsymbol{R}_{23}^{(23)}\xi_2\xi_3 + \boldsymbol{R}_{23}^{(24)}\xi_2\xi_4 + \boldsymbol{R}_{23}^{(34)}\xi_3\xi_4 + \boldsymbol{R}_{23}^{(1234)}\xi_1\xi_2\xi_3\xi_4. \end{aligned}\right\} \tag{12}$$

Substitute the given expressions into Eq.(3) and we will get the system of equations for components

$$\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(0)} = \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(0)}, \tag{13.1}$$

$$\begin{aligned} &\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(\alpha)} + \boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(\alpha)}\boldsymbol{R}_{23}^{(0)} + \boldsymbol{R}_{12}^{(\alpha)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(0)}\\ &= \boldsymbol{R}_{23}^{(\alpha)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(0)} + \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(\alpha)}\boldsymbol{R}_{12}^{(0)} + \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(\alpha)},\\ &\qquad \alpha = 12,13,14,23,24,34, \end{aligned} \tag{13.2}$$

$$\begin{aligned} &\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(12)}\boldsymbol{R}_{23}^{(34)} - \boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(13)}\boldsymbol{R}_{23}^{(24)} + \boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(14)}\boldsymbol{R}_{23}^{(23)} +\\ &\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(23)}\boldsymbol{R}_{23}^{(14)} - \boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(24)}\boldsymbol{R}_{23}^{(13)} + \boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(34)}\boldsymbol{R}_{23}^{(12)} +\\ &\boldsymbol{R}_{12}^{(12)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(34)} - \boldsymbol{R}_{12}^{(13)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(24)} + \boldsymbol{R}_{12}^{(14)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(23)} +\\ &\boldsymbol{R}_{12}^{(23)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(14)} - \boldsymbol{R}_{12}^{(24)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(13)} + \boldsymbol{R}_{12}^{(34)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(12)} +\\ &\boldsymbol{R}_{12}^{(12)}\boldsymbol{R}_{13}^{(34)}\boldsymbol{R}_{23}^{(0)} - \boldsymbol{R}_{12}^{(13)}\boldsymbol{R}_{13}^{(24)}\boldsymbol{R}_{23}^{(0)} + \boldsymbol{R}_{12}^{(14)}\boldsymbol{R}_{13}^{(23)}\boldsymbol{R}_{23}^{(0)} +\\ &\boldsymbol{R}_{12}^{(23)}\boldsymbol{R}_{13}^{(14)}\boldsymbol{R}_{23}^{(0)} - \boldsymbol{R}_{12}^{(24)}\boldsymbol{R}_{13}^{(13)}\boldsymbol{R}_{23}^{(0)} + \boldsymbol{R}_{12}^{(34)}\boldsymbol{R}_{13}^{(12)}\boldsymbol{R}_{23}^{(12)} +\\ &\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(1234)} + \boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(1234)}\boldsymbol{R}_{23}^{(0)} + \boldsymbol{R}_{12}^{(1234)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(0)} +\\ &= \boldsymbol{R}_{23}^{(1234)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(0)} + \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(1234)}\boldsymbol{R}_{12}^{(0)} + \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(1234)} +\\ &\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(12)}\boldsymbol{R}_{12}^{(34)} - \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(13)}\boldsymbol{R}_{12}^{(34)} + \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(14)}\boldsymbol{R}_{12}^{(23)} +\\ &\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(23)}\boldsymbol{R}_{12}^{(14)} - \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(24)}\boldsymbol{R}_{12}^{(13)} + \boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(34)}\boldsymbol{R}_{12}^{(12)} +\\ &\boldsymbol{R}_{23}^{(12)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(34)} - \boldsymbol{R}_{23}^{(13)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(34)} + \boldsymbol{R}_{23}^{(14)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(23)} +\\ &\boldsymbol{R}_{23}^{(23)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(14)} - \boldsymbol{R}_{23}^{(24)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(13)} + \boldsymbol{R}_{23}^{(34)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(12)} +\\ &\boldsymbol{R}_{23}^{(12)}\boldsymbol{R}_{13}^{(34)}\boldsymbol{R}_{12}^{(0)} - \boldsymbol{R}_{23}^{(13)}\boldsymbol{R}_{13}^{(24)}\boldsymbol{R}_{12}^{(0)} + \boldsymbol{R}_{23}^{(14)}\boldsymbol{R}_{13}^{(23)}\boldsymbol{R}_{12}^{(0)} +\\ &\boldsymbol{R}_{23}^{(23)}\boldsymbol{R}_{13}^{(14)}\boldsymbol{R}_{12}^{(0)} - \boldsymbol{R}_{23}^{(24)}\boldsymbol{R}_{13}^{(13)}\boldsymbol{R}_{12}^{(0)} + \boldsymbol{R}_{23}^{(34)}\boldsymbol{R}_{13}^{(12)}\boldsymbol{R}_{12}^{(0)}, \end{aligned} \tag{13.3}$$

where Eq.(13.1) represents the standard constant Yang-Baxter equation for matrices over a numerical field. All possible solutions of Eq.(13.1) were derived in (Hietarinta, 1992; 1993). In general, using these solutions by means of Eq.(13), all corresponding solutions classes over the Grassmann algebra can be obtained.

In the particular case, when there exists the symmetry

$$\boldsymbol{R}^{(12)}=\boldsymbol{R}^{(13)}=\boldsymbol{R}^{(14)}=\boldsymbol{R}^{(23)}=\boldsymbol{R}^{(24)}=\boldsymbol{R}^{(34)}=\boldsymbol{R}^{(1)},\tag{14}$$

then Eq.(13) simplifies and is represented in the following way

$$\left.\begin{aligned}
&\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(0)}=\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(0)},\\
&\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(1)}+\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(1)}\boldsymbol{R}_{23}^{(0)}+\boldsymbol{R}_{12}^{(1)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(0)}\\
&=\boldsymbol{R}_{23}^{(1)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(0)}+\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(1)}\boldsymbol{R}_{12}^{(0)}+\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(1)},\\
&2(\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(1)}\boldsymbol{R}_{23}^{(1)}+\boldsymbol{R}_{12}^{(1)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(1)}+\boldsymbol{R}_{12}^{(1)}\boldsymbol{R}_{13}^{(1)}\boldsymbol{R}_{23}^{(0)})+\\
&\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(1234)}+\boldsymbol{R}_{12}^{(0)}\boldsymbol{R}_{13}^{(1234)}\boldsymbol{R}_{23}^{(0)}+\boldsymbol{R}_{12}^{(1234)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{23}^{(0)}\\
&=2(\boldsymbol{R}_{23}^{(1)}\boldsymbol{R}_{13}^{(1)}\boldsymbol{R}_{12}^{(0)}+\boldsymbol{R}_{23}^{(1)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(1)}+\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(1)}\boldsymbol{R}_{12}^{(1)})+\\
&\boldsymbol{R}_{23}^{(1234)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(0)}+\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(1234)}\boldsymbol{R}_{12}^{(0)}+\boldsymbol{R}_{23}^{(0)}\boldsymbol{R}_{13}^{(0)}\boldsymbol{R}_{12}^{(1234)}.
\end{aligned}\right\}\tag{15}$$

Thus, there are solutions, satisfying the given system, but with non-zero nilpotent part.

From the formal point of view, purely nilpotent solutions [not containing numerical parts in Eq.(11)] exist with any number of Grassmann generators. For cubic nilpotent equation to be different from zero, it is necessary to have at least 6 generators of the Grassmann algebra. That is why for the number of generators less than 6 pure nilpotent solution [under $\boldsymbol{R}^{(0)}=0$] will be any matrix, as the equations for the components of $\boldsymbol{R}$-matrix are homogeneous of the third degree.

If we consider $\boldsymbol{R}$-matrix of the general type with all non-zero elements, the Yang-Baxter equation gives the system of 64 equations with 16 unknowns (Hietarinta, 1997). Considering expansions over the Grassmann algebra Eq.(11), the number of equations and the number of variables grow essentially. That is why we will consider only 6-vertex solutions.

NONINVERTIBLE SOLUTIONS OF SOME EQUATIONS OVER GRASSMANN ALGEBRA

Let us consider the set of equations, which arise in finding the matrix solutions of Yang-Baxter equation over the even part of Grassmann algebra

$$ax=0,\tag{16}$$

$$abc=0,\tag{17}$$

$$a^2=b^2.\tag{18}$$

Consider the case when a is invertible. Then multiplying the left side of Eq.(16) by a^{-1}, we get the only solution x=0. Analogous arguments are true for Eq.(17) too. From Eq.(18) we write the equation onto numerical parts a and b: $a_0^2=b_0^2$. From here it follows, that a and b are invertible and noninvertible simultaneously. That is why the solutions of this equation for every value of the numerical part can differ in the nilpotent part only. Let it be assumed that for every a there exist two solutions b and $b'=b+\tilde{b}$, where $\tilde{b}$ is nilpotent. Then $b^2=b^2+2b\tilde{b}+\tilde{b}^2$, from which it follows, that $\tilde{b}(2b+\tilde{b})=0$. The numerical part of the expression in brackets equals $2b_0$, i.e. it is invertible. That is why $\tilde{b}$=0. Consequently, the solution of Eq.(18) will be $b=\pm a$.

Thus, new solutions can only exist in the class of purely nilpotent a. That is why we will consider further only nilpotent elements included into Eq.(16).

For Eq.(16) it will be further shown that for any a we can construct x. Then for Eq.(17) the derived result can be used by substitution into Eq.(16) a over ab, and x over c. That is, for arbitrary ab we can construct c.

Eq.(18) by substitution $a=e+f$, $b=e-f$ can be transformed into the type of Eq.(16): ef=0.

In Eq.(16) for every nilpotent a we will find all possible b. For the solution of Eq.(16) with a given a, it is convenient to represent it in the form of products sum of the Grassmann algebra generators with invertible parameter. For further computations it will be useful to write the solutions of Eq.(16) for some character values of a.

Type 1: a is represented by one term

$$a=\xi_{i_1}\cdots\xi_{i_n}\hat{a},\hat{a}_0\neq 0.$$

As $\tilde{a}$ is invertible, then Eq.(16) is equivalent to the equation

$$\xi_{i_1}\cdots\xi_{i_n}x=0.\tag{19}$$

For the solution of this equation, let us present x in the form of the sum of all possible ordered products of elements. Then in Eq.(19) the terms without ξ_i will remain. As the product of generators included into x are ordered, then under each product of generators in Eq.(19) there will be only one numerical term. That is why the general solution of Eq.(19) will be

$$x = \sum_{k=1}^{n} \xi_{i_k} \widehat{x}_k ,$$

where $\tilde{x}_k$ are arbitrary, possibly invertible parameters of the Grassmann algebra. Then it is necessary that x contains at least one term included into $\xi_{i_1} \ldots \xi_{i_n}$.

Type 2: variable a has the following presentation

$$a = \xi_{i_1} \cdots \xi_{i_l} a_1 + \xi_{j_1} \cdots \xi_{j_m} a_2 , \tag{20}$$

There are sets of indices, which do not intersect $\{i_1,\ldots,i_l\}\cap\{j_1,\ldots,j_m\}=\varnothing$. Without loss of generality it can be assumed that $a_2=1$. Then x can be conveniently presented in the form of two terms $x=x_1+x_2$.

In the first of them two terms $\xi_{i_1} \cdots \xi_{i_l} x_1 = 0$ and $\xi_{j_1} \cdots \xi_{j_m} x_1 = 0$ vanish. Then using the result of the previous case, we get

$$x_1 = \sum_{p=1}^{l} \sum_{q=1}^{m} \xi_{i_p} \xi_{j_q} x_{pq} ,$$

where x_{pq} are arbitrary elements.

The variable x_2 will be defined from the condition that x_2 is not zero, when at least one of the terms in Eq.(20) does not vanish. For the sum to be nonzero, as $\{i_1,\ldots,i_l\}\cap\{j_1,\ldots,j_m\}=\varnothing$, it is necessary for x_2 to be

$$x_2 = \xi_{j_1} \ldots \xi_{j_m} x_3 + \xi_{i_1} \cdots \xi_{i_l} x_4 , \tag{21}$$

where x_3 and x_4 are arbitrary parameters which do not contain $\xi_{i_1},\cdots,\xi_{i_l}$, $\xi_{j_1},\cdots,\xi_{j_m}$. Then Eq.(16) is equivalent to the equation

$$a_1' x_3 + (-1)^{ml} x_4 = 0,$$

where $\xi_{j_1} \cdots \xi_{j_m} a_1' = a_1 \xi_{j_1} \cdots \xi_{j_m}$ and a_1' does not contain terms containing $\xi_{j_1},\cdots,\xi_{j_m}$. Then

$$\boldsymbol{x}_2 = (\xi_{j_1} \cdots \xi_{j_m} + \xi_{i_1} \cdots \xi_{i_l} (-1)^{ml+1} a_1') \boldsymbol{x}_3 .$$

Type 3: a is presented in the form of two terms, as in the previous case, except that sets of the indices intersect $\{i_1,\ldots,i_l\}\cap\{j_1,\ldots,j_m\}\neq\varnothing$. Therefore a can be written in the following form

$$a = \xi_{i_1} \cdots \xi_{i_l} (\xi_{j_1} \cdots \xi_{j_m} a_1 + \xi_{k_1} \cdots \xi_{k_n}), \tag{22}$$

$\{i_1,\ldots,i_l\}\cap\{j_1,\ldots,j_m\}=\varnothing$, $\{i_1,\ldots,i_l\}\cap\{k_1,\ldots,k_n\}=\varnothing$, $\{j_1,\ldots,j_m\}\cap\{k_1,\ldots,k_n\}=\varnothing$. Then parameter x can be presented in the form of three terms

$$x=x_1+x_2+x_3 , \tag{23}$$

In the first term $\xi_{i_1} \cdots \xi_{i_l}$ vanishes, i.e.

$$\xi_{i_1} \cdots \xi_{i_l} x_1 = 0.$$

Then using the result of the first case, we get

$$x_1 = \sum_{p=1}^{l} \xi_{i_p} \widehat{x}_p .$$

In the second term both terms in the bracket vanish simultaneously, i.e. Eq.(22)

$$\xi_{j_1} \cdots \xi_{j_m} x_2 = 0 \wedge \xi_{k_1} \cdots \xi_{k_n} x_2 = 0 .$$

Then in analogy with the previous case

$$x_2 = \sum_{p=1}^{m} \sum_{q=1}^{n} \xi_{j_p} \xi_{k_q} x_{pq} .$$

In the third term the multiplier in the bracket vanishes as in the previous case. The type x_3 looks similar to Eq.(21)

$$x_3 = \xi_{k_1} \cdots \xi_{k_n} x_4 + \xi_{j_1} \cdots \xi_{j_m} x_5 ,$$

where x_4 and x_5 do not contain $\xi_{j_1},\cdots,\xi_{j_m}$, $\xi_{k_1},\cdots,\xi_{k_n}$. Then Eq.(16) is equivalent to the equation

$$a_1' x_4 + (-1)^{ml} x_5 = 0,$$

where $\xi_{k_1} \cdots \xi_{k_n} a_1' = a_1 \xi_{k_1} \cdots \xi_{k_n}$ and a_1' does not have terms containing $\xi_{k_1},\cdots,\xi_{k_n}$. Then, finally

$$x_3=(\xi_{j_1}\cdots\xi_{j_m}+\xi_{i_1}\cdots\xi_{i_l}(-1)^{ml+1}a_1')x_4.$$

Using the obtained results we can construct the general type of solution for Eq.(16) under the given type a

$$a=\sum_{k=1}^{n}\alpha_k a_k, \tag{24}$$

where $\alpha_k=\xi_{i_1}\cdots\xi_{i_l}$, $i_1,\cdots,i_l$ are in increasing order, a_k are invertible parameters not containing $\xi_{i_1}\cdots\xi_{i_l}$. The general process of finding solutions is as follows: let us present x in the form of the sum n+1 of terms, where n is the number of terms in Eq.(24)

$$x=\sum_{i=0}^{n}x_i.$$

Every x_i vanishes simultaneously in the i terms in Eq.(24), that such terms are chosen as any, i.e., each x_i is the sum $x_i=\sum_{j=1}^{\binom{n}{i}}x_{ij}$, in each term of which vanishes some concrete selection C_{ij} from n over i terms in Eq.(24)

$$\forall k\in C_{ij},\ a_k x_{ij}=0,\ \sum_{k\notin C_j}a_k x_{ij}=0. \tag{25}$$

Cases j=n and j=n−1 coincide, as all the sum vanishes and the sum of n−1 of its terms also vanishes, then nth term vanishes from each term.

The situation can be such that for some selection C_{ij}, some terms will vanish in the sum in Eq.(25). Then such case will become the case with greater i. That is why it is necessary to begin with greater i in the search for x. The same order of finding x is defined by the way of finding the solutions for quotient types a.

Thus, the given scheme allows getting all possible solutions of Eq.(16).

Consider the examples. Let us find some concrete even nilpotent equation solutions of Eq.(16) over Grassmann algebra with 2,3,4,5 elements.

Case 1: For elements of the type

$$a=a_{12}\xi_1\xi_2,\ b=b_{12}\xi_1\xi_2,$$

Eq.(16) is satisfied $\forall a_{12}$, b_{12}, as ξ_i^2=0. Identical situation is true for 3 elements

$$\begin{aligned}a&=a_{12}\xi_1\xi_2+a_{13}\xi_1\xi_3+a_{23}\xi_2\xi_3,\\ b&=b_{12}\xi_1\xi_2+b_{13}\xi_1\xi_3+b_{23}\xi_2\xi_3,\end{aligned}$$

and all the coefficients are arbitrary.

Case 2: For elements of the type

$$\begin{aligned}a=\ &a_{12}\xi_1\xi_2+a_{13}\xi_1\xi_3+a_{14}\xi_1\xi_4+a_{23}\xi_2\xi_3+\\ &a_{24}\xi_2\xi_4+a_{34}\xi_3\xi_4+a_{1234}\xi_1\xi_2\xi_3\xi_4,\\ b=\ &b_{12}\xi_1\xi_2+b_{13}\xi_1\xi_3+b_{14}\xi_1\xi_4+b_{23}\xi_2\xi_3+\\ &b_{24}\xi_2\xi_4+b_{34}\xi_3\xi_4+b_{1234}\xi_1\xi_2\xi_3\xi_4,\end{aligned}$$

Eq.(16) comes to the following expression for the coefficients

$$a_{12}b_{34}-a_{13}b_{24}+a_{14}b_{23}+a_{23}b_{14}-a_{24}b_{13}+a_{34}b_{12}=0,$$

but a_{1234}, b_{1234} remain undefined. Thus, the solution will be 13-parametrical. Analogous situation exists for 5 elements, then

$$\begin{aligned}a=\ &a_{12}\xi_1\xi_2+a_{13}\xi_1\xi_3+a_{14}\xi_1\xi_4+a_{15}\xi_1\xi_5+\\ &a_{23}\xi_2\xi_3+a_{24}\xi_2\xi_4+a_{25}\xi_2\xi_5+a_{34}\xi_3\xi_4+\\ &a_{35}\xi_3\xi_5+a_{45}\xi_4\xi_5+a_{1234}\xi_1\xi_2\xi_3\xi_4+a_{1235}\xi_1\xi_2\xi_3\xi_5+\\ &a_{1245}\xi_1\xi_2\xi_4\xi_5+a_{1345}\xi_1\xi_3\xi_4\xi_5+a_{2345}\xi_2\xi_3\xi_4\xi_5,\\ b=\ &b_{12}\xi_1\xi_2+b_{13}\xi_1\xi_3+b_{14}\xi_1\xi_4+b_{15}\xi_1\xi_5+\\ &b_{23}\xi_2\xi_3+b_{24}\xi_2\xi_4+b_{25}\xi_2\xi_5+b_{34}\xi_3\xi_4+\\ &b_{35}\xi_3\xi_5+b_{45}\xi_4\xi_5+b_{1234}\xi_1\xi_2\xi_3\xi_4+b_{1235}\xi_1\xi_2\xi_3\xi_5+\\ &b_{1245}\xi_1\xi_2\xi_4\xi_5+b_{1345}\xi_1\xi_3\xi_4\xi_5+b_{2345}\xi_2\xi_3\xi_4\xi_5.\end{aligned}$$

Eq.(16) gives the system

$$\begin{aligned}&a_{12}b_{34}-a_{13}b_{24}+a_{14}b_{23}+a_{23}b_{14}-a_{24}b_{13}+a_{34}b_{12}=0,\\ &a_{12}b_{35}-a_{13}b_{25}+a_{15}b_{23}+a_{23}b_{15}-a_{25}b_{13}+a_{35}b_{12}=0,\\ &a_{12}b_{45}-a_{14}b_{25}+a_{15}b_{24}+a_{24}b_{15}-a_{25}b_{14}+a_{45}b_{12}=0,\\ &a_{13}b_{45}-a_{14}b_{35}+a_{15}b_{34}+a_{34}b_{15}-a_{35}b_{14}+a_{45}b_{13}=0,\\ &a_{23}b_{45}-a_{24}b_{35}+a_{25}b_{34}+a_{34}b_{25}-a_{35}b_{24}+a_{45}b_{23}=0,\end{aligned}$$

where coefficients a_{1234}, a_{1235}, a_{1245}, a_{1345}, a_{2345}, b_{1234}, b_{1235}, b_{1245}, b_{1345}, b_{2345} are undefined. The solution will be 25-parametrical. It is easy to see that in the case of 6 elements there will be equations on coefficients of the new type.

Thus, it can be assumed that the solutions of Eq.(16) over Grassmann algebra with $2n$ and $2n+1$ generators are defined by the same conditions.

Then we consider the Yang-Baxter equation solution over Grassmann algebra.

CLASSIFICATION OF YANG-BAXTER EQUATION SOLUTIONS WITH THE NUMBER OF VERTICES, LESS OR EQUAL TO SIX

According to (Kassel, 1995), 6-vertex solution for the Yang-Baxter equation is $\boldsymbol{R}$-matrix of the type

$$\boldsymbol{R}=\begin{pmatrix} p & \cdot & \cdot & \cdot \\ \cdot & c & d & \cdot \\ \cdot & a & b & \cdot \\ \cdot & \cdot & \cdot & q \end{pmatrix}, \tag{26}$$

where a, b, c, d, p, q are even elements of the Grassmann algebra. From Yang-Baxter Eq.(3) it follows, that $\boldsymbol{R}$-matrix is defined precisely up to a constant (scale symmetry), so that it is always possible to define the normalization of the element, whose numerical part differs from zero. As it is unknown beforehand (before classification), which of the elements in Eq.(3) has non-zero numerical part, we are not going to do the normalization Eq.(26).

From Eq.(2) and Eq.(3), the evident type of matrix $\boldsymbol{R}_{12}$, $\boldsymbol{R}_{13}$, $\boldsymbol{R}_{23}$ leads to

$$\boldsymbol{R}_{12}=\begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c & \cdot & d & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \end{pmatrix},$$

$$\boldsymbol{R}_{13}=\begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c & \cdot & \cdot & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c & \cdot & \cdot & d & \cdot \\ \cdot & a & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot \\ \cdot & \cdot & \cdot & a & \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \end{pmatrix},$$

$$\boldsymbol{R}_{23}=\begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c & d & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & d & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \end{pmatrix}. \tag{27}$$

Let us substitute $\boldsymbol{R}_{12}$, $\boldsymbol{R}_{13}$, $\boldsymbol{R}_{23}$ into Eq.(6.1) and get the following (essentially redefined) system of equations

$$cda=0, \tag{28}$$

$$bda=0, \tag{29}$$

$$da(d-a)=0,\ pd(d-p)+cbd=0, \tag{30}$$

$$\left.\begin{aligned} qd(d-q)+cbd=0,\\ pa(a-p)+cba=0,\\ qa(a-q)+cba=0. \end{aligned}\right\} \tag{31}$$

Let us present all the parameters in the form of sum of numerical part and even nilpotent part as $x=x_0+\tilde{x}$, where $x=a,b,c,d,p,q$. It is clear that Eqs.(28)~(31) must be separately satisfied by the numerical part, so that the classification will be done according to it.

Because in Eq.(28) $cda=0$, it is convenient to classify the solutions for the numerical part with vanishing a_0 and d_0.

(1) Both a_0 and d_0 differ from zero: d_0, $a_0 \neq 0 \rightarrow b_0=c_0=0$, $a_0=d_0$, $\{p_0, q_0\}=\{0, a_0\}$, where $\{\}$ means the set of elements. Then for the numerical part of $\boldsymbol{R}$-matrix we derive 4-vertex solution

$$\boldsymbol{R}^{(0)}=\begin{pmatrix} \{0,a_0\} & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_0 & \cdot \\ \cdot & a_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \{0,a_0\} \end{pmatrix}\sim\begin{pmatrix} \{0,1\} & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \{0,1\} \end{pmatrix}, \tag{32}$$

where the last equivalency follows from the normalization because $a_0 \neq 0$.

(2) One of the elements a_0 or d_0 differs from zero:

(a) $a_0=0$,

$$d_0 \neq 0 \rightarrow \{p_0,q_0\}=\left\{\frac{d_0}{2}\pm\sqrt{\frac{d_0^2}{4}+b_0c_0}\right\},$$

then for the numerical part of $\boldsymbol{R}$-matrix we get 5-vertex solution

$$\boldsymbol{R}^{(0)}=\begin{pmatrix} \frac{d_0}{2}\pm\sqrt{\frac{d_0^2}{4}+b_0c_0} & \cdot & \cdot & \cdot \\ \cdot & c_0 & d_0 & \cdot \\ \cdot & \cdot & b_0 & \cdot \\ \cdot & \cdot & \cdot & \frac{d_0}{2}\pm\sqrt{\frac{d_0^2}{4}+b_0c_0} \end{pmatrix}. \tag{33}$$

(b) Or for $d_0=0, a_0\neq 0 \rightarrow \{p_0,q_0\}=\{a_0/2\pm\sqrt{a_0^2/4+b_0c_0}\}$, then again we get 5-vertex solution

$$\boldsymbol{R}^{(0)}=\begin{pmatrix} \frac{a_0}{2}\pm\sqrt{\frac{a_0^2}{4}+b_0c_0} & \cdot & \cdot & \cdot \\ \cdot & c_0 & \cdot & \cdot \\ \cdot & a_0 & b_0 & \cdot \\ \cdot & \cdot & \cdot & \frac{a_0}{2}\pm\sqrt{\frac{a_0^2}{4}+b_0c_0} \end{pmatrix}. \tag{34}$$

(3) Both numerical parts of the elements a and d equal zero: $a_0=d_0=0 \rightarrow p_0$, q_0, b_0, c_0 are any, and the numerical part of $\boldsymbol{R}$-matrix becomes diagonal (4-vertex numerical solution)

$$\boldsymbol{R}^{(0)}=\begin{pmatrix} p_0 & \cdot & \cdot & \cdot \\ \cdot & c_0 & \cdot & \cdot \\ \cdot & \cdot & b_0 & \cdot \\ \cdot & \cdot & \cdot & q_0 \end{pmatrix}. \tag{35}$$

Now let us continue the classification based on nilpotent parts. Using the fact that the element is invertible, if its numerical part differs from zero, we get the final answer for the first case:

(1) $a=d$ is any even invertible: $b=c=0$, $\{p, q\}=\{0, a\}$, then we have 2- and 4-vertex solutions

$$\boldsymbol{R}=\begin{pmatrix} \{0,a\} & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot \\ \cdot & a & \cdot & \cdot \\ \cdot & \cdot & \cdot & \{0,a\} \end{pmatrix}\sim\begin{pmatrix} \{0,1\} & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \{0,1\} \end{pmatrix}, \tag{36}$$

where the last equivalency is derived from Yang-Baxter equation scale symmetry.

In the second case the invertibility d leads to the vanishing of nilpotent parameter a from Eq.(28). Then the remaining simplified system of Eqs.(30)~(31) can be solved according to parameters p and q. Equations on these parameters coincide

$$p^2=pd+bc,\ q^2=qd+bc\,. \tag{37}$$

If the quadratic equation discriminant has a nonzero numerical part $d_0^2/4+b_0c_0\neq 0$, then the solution can be written in the following way.

(2) Under $a=0$ we have:

(a) $d_0^2/4+b_0c_0\neq 0 \rightarrow \{p,q\}=d/2\pm\sqrt{d^2/4+bc}$, then we get 5-vertex solution

$$\boldsymbol{R}=\begin{pmatrix} \frac{d}{2}\pm\sqrt{\frac{d^2}{4}+bc} & \cdot & \cdot & \cdot \\ \cdot & c & d & \cdot \\ \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \frac{d}{2}\pm\sqrt{\frac{d^2}{4}+bc} \end{pmatrix}. \tag{38}$$

If the discriminant is equal to zero, i.e., $d_0^2/4+b_0c_0=0$, then Eq.(38) is useless, as it is impossible to find the square root from nilpotent expression. Then from Eqs.(28)~(31) and Eq.(37) it is seen that the vanishing discriminant numerical part leads to invertibility of parameters b, c, p, q. Suppose (using scale symmetry) d=1. The equations on the remaining nilpotent parts p and q are given as

$$\tilde{p}^2 = \tilde{q}^2. \tag{39}$$

Then we get:

(b) $d_0^2/4+b_0c_0=0 \rightarrow b$, c, p, q is invertible and $p_0 = q_0 = d_0/2, \quad c = (\tilde{p}^2 - 1/4)/b, \quad b_0 c_0 = -1/4$, p_0=q_0=1/2.

So we get again 5-vertex solution

$$\boldsymbol{R} = \begin{pmatrix} 1/2+\tilde{p} & \cdot & \cdot & \cdot \\ \cdot & \dfrac{\tilde{p}^2-1/4}{b} & 1 & \cdot \\ \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & 1/2+\tilde{q} \end{pmatrix}. \tag{40}$$

In this case we have the set of solutions in comparison with the previous case due to nilpotency. The main difference from the previous case Eq.(38) is that the parameters p and q are now connected by Eq.(39) on nilpotent parts, which gives great freedom in choosing their meanings.

(3) In the third case, when the numerical part has a diagonal form, we can point out four sub-cases.

(a) $p_0^2-b_0c_0 \neq 0$ or $q_0^2-b_0c_0 \neq 0 \rightarrow a=d=0$, p,q,b,c are any, then we get diagonal $\boldsymbol{R}$-matrix (4-vertex solution)

$$\boldsymbol{R} = \begin{pmatrix} p & \cdot & \cdot & \cdot \\ \cdot & c & \cdot & \cdot \\ \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & q \end{pmatrix}. \tag{41}$$

(b) $p_0^2-b_0c_0=0$ and $q_0^2-b_0c_0=0$. Suppose p=1, then $p_0^2=q_0^2=b_0c_0=1$,

$$ad=0, \tag{42}$$

$$a\left(a+\frac{\tilde{c}}{c_0}+\frac{\tilde{b}}{b_0}+\frac{\tilde{b}}{b_0}\frac{\tilde{c}}{c_0}\right)=0, \tag{43}$$

$$d\left(d+\frac{\tilde{c}}{c_0}+\frac{\tilde{b}}{b_0}+\frac{\tilde{b}}{b_0}\frac{\tilde{c}}{c_0}\right)=0. \tag{44}$$

The numerical part of the parameter q because of q_0^2=1 can have the meaning of ±1. Consequently the equations for the nilpotent part $\tilde{q}$ lead to two different systems of equations

$$q_0 = +1 \rightarrow a\tilde{q}=0, d\tilde{q}=0, \tag{45}$$

$$q_0 = -1 \rightarrow a(a-\tilde{q})=0, d(d-\tilde{q})=0. \tag{46}$$

We rewrite Eq.(46), using Eqs.(43)~(44), in linear form over a and d as

$$\left.\begin{aligned} q_0 &= +1 \rightarrow a\tilde{q}=0, d\tilde{q}=0, \\ q_0 &= -1 \rightarrow a(\tilde{q}-f)=0, d(\tilde{q}-f)=0. \end{aligned}\right\} \tag{47}$$

where $f=-\dfrac{\tilde{b}}{b_0}-\dfrac{\tilde{c}}{c_0}-\dfrac{\tilde{b}}{b_0}\dfrac{\tilde{c}}{c_0}$.

(c) $p_0^2-b_0c_0=0$ and $q_0^2-b_0c_0=0 \rightarrow p_0=q_0=b_0=0$, $c_0 \neq 0 \rightarrow c=1$ and Eqs.(42)~(44) have the following form

$$\left.\begin{aligned} &ad=0, \\ &a(p^2-pa-b)=0,\ d(p^2-pd-b)=0, \\ &a(q^2-qa-b)=0,\ d(q^2-qd-b)=0. \end{aligned}\right\} \tag{48}$$

(d) All numerical parts are equal to zero: p_0=q_0=b_0=c_0=a_0=d_0=0.

Eqs.(42)~(44) and Eq.(48) reveal that only in Cases 3(b) and 3(c) we can expect full 6-vertex solutions, which are absent in the classification (Hietarinta, 1993; 1997).

Now let us consider the concrete type of solutions when the number of Grassmann algebra generators is 2, 3, and 4.

YANG-BAXTER EQUATION SOLUTION OVER GRASSMANN ALGEBRA WITH TWO ELEMENTS

We present the expansion of each element of $\boldsymbol{R}$-matrix Eq.(26) over 2 Grassmann generators as

$$c = c_0 + c_{12}\xi_1\xi_2,\ b = b_0 + b_{12}\xi_1\xi_2,\ d = d_0 + d_{12}\xi_1\xi_2,$$
$$p = p_0 + p_{12}\xi_1\xi_2,\ q = q_0 + q_{12}\xi_1\xi_2,\ a = a_0 + a_{12}\xi_1\xi_2. \quad (49)$$

Substitution of these equations into the system of Eqs.(28)~(31) yields the system of equations below

$$c_0d_0a_0 = 0,\ \ b_0d_0a_0 = 0,\ \ d_0a_0(d_0 - a_0) = 0, \quad (50)$$

$$\begin{aligned}
&p_0d_0(d_0 - p_0) + c_0b_0d_0 = 0,\ \ q_0d_0(d_0 - q_0) + c_0b_0d_0 = 0,\\
&p_0a_0(a_0 - p_0) + c_0b_0a_0 = 0,\ q_0a_0(a_0 - q_0) + c_0b_0a_0 = 0,\\
&c_0d_0a_{12} + c_0a_0d_{12} + d_0a_0c_{12} = 0,\\
&b_0d_0a_{12} + b_0a_0d_{12} + d_0a_0b_{12} = 0,\\
&d_0a_0(d_{12} - a_{12}) + d_0(d_0 - a_0)a_{12} + a_0(d_0 - a_0)d_{12} = 0,\\
&p_0d_0(d_{12} - p_{12}) + p_0(d_0 - p_0)d_{12} +\\
&d_0(d_0 - p_0)p_{12} + c_0b_0d_{12} + c_0d_0b_{12} + b_0d_0c_{12} = 0,\\
&q_0d_0(d_{12} - q_{12}) + q_0(d_0 - q_0)d_{12} + d_0(d_0 - q_0)q_{12} +\\
&c_0b_0d_{12} + c_0d_0b_{12} + b_0d_0c_{12} = 0,\\
&p_0a_0(a_{12} - p_{12}) + p_0(a_0 - p_0)a_{12} + a_0(a_0 - p_0)p_{12} +\\
&c_0b_0a_{12} + c_0a_0b_{12} + b_0a_0c_{12} = 0,\\
&q_0a_0(a_{12} - q_{12}) + q_0(a_0 - q_0)a_{12} + a_0(a_0 - q_0)q_{12} +\\
&c_0b_0a_{12} + c_0a_0b_{12} + b_0a_0c_{12} = 0.
\end{aligned} \quad (51)$$

Eq.(50) will be considered as the key ones. We find the solutions for all non-zero elements c, b, d, p, q, a as numerical and nilpotent parts do not vanish simultaneously, which essentially narrows the solutions classes.

Case I: d_0=0. Eqs.(50)~(51) have the following form:

$$\left.\begin{aligned}&d_0 = 0,\ \ p_0a_0(a_0 - p_0) + c_0b_0a_0 = 0,\\ &q_0a_0(a_0 - q_0) + c_0b_0a_0 = 0,\end{aligned}\right\} \quad (52.1)$$

$$a_0^2d_{12} = 0, \quad (52.2)$$

$$c_0a_0d_{12} = 0,\ \ b_0a_0d_{12} = 0, \quad (52.3)$$

$$p_0^2d_{12} - c_0b_0d_{12} = 0,\ \ q_0^2d_{12} - c_0b_0d_{12} = 0, \quad (52.4)$$

$$\begin{aligned}&p_0a_0(a_{12} - p_{12}) + p_0(a_0 - p_0)a_{12} + a_0(a_0 - p_0)p_{12} +\\ &c_0b_0a_{12} + c_0a_0b_{12} + b_0a_0c_{12} = 0,\end{aligned} \quad (52.5)$$

$$\begin{aligned}&q_0a_0(a_{12} - q_{12}) + q_0(a_0 - q_0)a_{12} + a_0(a_0 - q_0)q_{12} +\\ &c_0b_0a_{12} + c_0a_0b_{12} + b_0a_0c_{12} = 0.\end{aligned} \quad (52.6)$$

From Eq.(52.2) it follows, that a_0=0 and

$$\left.\begin{aligned}&d_0 = 0,\ a_0 = 0,\\ &p_0^2d_{12} - c_0b_0d_{12} = 0,\ q_0^2d_{12} - c_0b_0d_{12} = 0,\\ &p_0^2a_{12} - c_0b_0a_{12} = 0,\ q_0^2a_{12} - c_0b_0a_{12} = 0.\end{aligned}\right\} \quad (53)$$

From Eq.(53) it follows, that $p_0^2=q_0^2=c_0b_0$ (because of the assumption, none of the 6 elements equals zero). Insert the parameters $p_0=q_0=t$, $c_0=r\neq0$, then $b_0=t^2/r$, as nilpotent parts are not defined from Eq.(53), insert 6 more parameters $c_{12}=v$, $b_{12}=w$, $d_{12}=y$, $p_{12}=l$, $q_{12}=m$, $a_{12}=n$. The final solution will be an 8-parametric one

$$\left.\begin{aligned}&c = r + v\xi_1\xi_2,\ b = t^2/r + w\xi_1\xi_2,\ d = y\xi_1\xi_2,\\ &p = t + l\xi_1\xi_2,\ q = t + m\xi_1\xi_2,\ a = n\xi_1\xi_2.\end{aligned}\right\} \quad (54)$$

Let us write the resulting 6-vertex $\boldsymbol{R}$-matrix

$$\boldsymbol{R} = \begin{pmatrix} t + l\xi_1\xi_2 & 0 & 0 & 0\\ 0 & r + v\xi_1\xi_2 & y\xi_1\xi_2 & 0\\ 0 & n\xi_1\xi_2 & t^2/r + w\xi_1\xi_2 & 0\\ 0 & 0 & 0 & t + m\xi_1\xi_2\end{pmatrix}$$
$$= \begin{pmatrix} t & 0 & 0 & 0\\ 0 & r & 0 & 0\\ 0 & 0 & t^2/r & 0\\ 0 & 0 & 0 & t\end{pmatrix} + \begin{pmatrix} l & 0 & 0 & 0\\ 0 & v & y & 0\\ 0 & n & w & 0\\ 0 & 0 & 0 & m\end{pmatrix}\xi_1\xi_2. \quad (55)$$

According to the classification of the previous section, such solution refers to Case 3(b). Evidently, if we considered $\boldsymbol{R}$-matrix over numerical field, this solution would be 4-vertex, but not 6 (Hietarinta, 1997). Under t=0 we obtain 6-vertex $\boldsymbol{R}$-matrix, in which only 1 element is invertible, that in our classification corresponds to Case 3(c).

Case II: a_0=0. Eqs.(50)~(51) lead to the following:

$$a_0 = 0, \quad (56.1)$$

$$\left.\begin{aligned}&p_0d_0(d_0 - p_0) + c_0b_0d_0 = 0,\\ &q_0d_0(d_0 - q_0) + c_0b_0d_0 = 0,\end{aligned}\right\} \quad (56.2)$$

$$c_0d_0a_{12} = 0,\ \ b_0d_0a_{12} = 0, \quad (56.3)$$

$$d_0^2a_{12} = 0, \quad (56.4)$$

$$p_0d_0(d_{12} - p_{12}) + p_0(d_0 - p_0)d_{12} + d_0(d_0 - p_0)$$

$$\times p_{12}+c_0b_0d_{12}+c_0d_0b_{12}+b_0d_0c_{12}=0, \quad (56.5)$$

$$q_0d_0(d_{12}-q_{12})+q_0(d_0-q_0)d_{12}+d_0(d_0-q_0)q_{12}+ c_0b_0d_{12}+c_0d_0b_{12}+b_0d_0c_{12}=0, \quad (56.6)$$

$$p_0^2a_{12}-c_0b_0a_{12}=0,\ \ q_0^2a_{12}-c_0b_0a_{12}=0. \quad (56.7)$$

From Eq.(56.4) it follows, that d_0=0, then Eq.(56) becomes to Eq.(53) and the solutions coincide with Case I.

Case III: c_0=b_0=0, d_0=a_0. Then Eqs.(50)~51) are as follows:

$$c_0=0,\ b_0=0,\ d_0=a_0, \quad (57.1)$$

$$p_0d_0(d_0-p_0)=0,\ q_0d_0(d_0-q_0)=0, \quad (57.2)$$

$$d_0a_0c_{12}=0,\ d_0a_0b_{12}=0, \quad (57.3)$$

$$d_0a_0(d_{12}-a_{12})=0, \quad (57.4)$$

$$p_0d_0(d_{12}-p_{12})+p_0(d_0-p_0)d_{12} +d_0(d_0-p_0)p_{12}=0, \quad (57.5)$$

$$q_0d_0(d_{12}-q_{12})+q_0(d_0-q_0)d_{12} +d_0(d_0-q_0)q_{12}=0. \quad (57.6)$$

Eq.(57.3) shows that a_0=b_0=c_0=d_0=0, then the coefficients p_0, q_0, a_{12}, b_{12}, c_{12}, d_{12}, p_{12}, q_{12} remain undefined. The solution will also be 8-parametrical p_0=t, q_0=r, a_{12}=l, b_{12}=m, c_{12}=n, d_{12}=k, p_{12}=w, q_{12}=s and can be represented in the following form

$$\boldsymbol{R}=\begin{pmatrix} t+w\xi_1\xi_2 & 0 & 0 & 0\\ 0 & n\xi_1\xi_2 & k\xi_1\xi_2 & 0\\ 0 & l\xi_1\xi_2 & m\xi_1\xi_2 & 0\\ 0 & 0 & 0 & r+s\xi_1\xi_2 \end{pmatrix} =\begin{pmatrix} t&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&r\end{pmatrix}+\begin{pmatrix} w&0&0&0\\0&n&k&0\\0&l&m&0\\0&0&0&s\end{pmatrix}\xi_1\xi_2. \quad (58)$$

This solution is 6-vertex $\boldsymbol{R}$-matrix with 2 invertible elements, which is not contained in the Hietarinta classification (Hietarinta, 1992; 1993).

SOLUTIONS OF YANG-BAXTER EQUATIONS OVER GRASSMAN ALGEBRA WITH THREE GENERATORS

Let us write the expansion of each element of $\boldsymbol{R}$-matrix Eq.(26) over 3 generators as

$$\begin{aligned} c&=c_0+c_{12}\xi_1\xi_2+c_{13}\xi_1\xi_3+c_{23}\xi_2\xi_3,\\ b&=b_0+b_{12}\xi_1\xi_2+b_{13}\xi_1\xi_3+b_{23}\xi_2\xi_3,\\ d&=d_0+d_{12}\xi_1\xi_2+d_{13}\xi_1\xi_3+d_{23}\xi_2\xi_3,\\ p&=p_0+p_{12}\xi_1\xi_2+p_{13}\xi_1\xi_3+p_{23}\xi_2\xi_3,\\ q&=q_0+q_{12}\xi_1\xi_2+q_{13}\xi_1\xi_3+q_{23}\xi_2\xi_3,\\ a&=a_0+a_{12}\xi_1\xi_2+a_{13}\xi_1\xi_3+a_{23}\xi_2\xi_3. \end{aligned} \quad (59)$$

Substituting Eq.(59) into Eqs.(28)~(31) yields the equations in the components

$$c_0d_0a_0=0,\ b_0d_0a_0=0,\ d_0a_0(d_0-a_0)=0, \quad (60)$$

$$\left.\begin{aligned} p_0d_0(d_0-p_0)+c_0b_0d_0&=0,\\ q_0d_0(d_0-q_0)+c_0b_0d_0&=0,\\ p_0a_0(a_0-p_0)+c_0b_0a_0&=0,\\ q_0a_0(a_0-q_0)+c_0b_0a_0&=0,\\ c_0d_0a_{12}+c_0a_0d_{12}+d_0a_0c_{12}&=0,\\ c_0d_0a_{13}+c_0a_0d_{13}+d_0a_0c_{13}&=0,\\ c_0d_0a_{23}+c_0a_0d_{23}+d_0a_0c_{23}&=0,\\ b_0d_0a_{12}+b_0a_0d_{12}+d_0a_0b_{12}&=0,\\ b_0d_0a_{13}+b_0a_0d_{13}+d_0a_0b_{13}&=0,\\ b_0d_0a_{23}+b_0a_0d_{23}+d_0a_0b_{23}&=0 \end{aligned}\right\} \quad (61)$$

$$\begin{aligned} d_0a_0(d_{12}-a_{12})+d_0(d_0-a_0)a_{12}+a_0(d_0-a_0)d_{12}&=0,\\ d_0a_0(d_{13}-a_{13})+d_0(d_0-a_0)a_{13}+a_0(d_0-a_0)d_{13}&=0,\\ d_0a_0(d_{23}-a_{23})+d_0(d_0-a_0)a_{23}+a_0(d_0-a_0)d_{23}&=0. \end{aligned} \quad (62)$$

and

$$\left.\begin{aligned} &p_0d_0(d_\beta-p_\beta)+p_0(d_0-p_0)d_\beta+d_0(d_0-p_0)p_\beta+\\ &\quad c_0b_0d_\beta+c_0d_0b_\beta+b_0d_0c_\beta=0,\\ &q_0d_0(d_\gamma-q_\gamma)+q_0(d_0-q_0)d_\gamma+d_0(d_0-q_0)q_\gamma+\\ &\quad c_0b_0d_\gamma+c_0d_0b_\gamma+b_0d_0c_\gamma=0,\\ &p_0a_0(a_\delta-p_\delta)+p_0(a_0-p_0)a_\delta+a_0(a_0-p_0)p_\delta+\\ &\quad c_0b_0a_\delta+c_0a_0b_\delta+b_0a_0c_\delta=0,\\ &q_0a_0(a_\varepsilon-q_\varepsilon)+q_0(a_0-q_0)a_\varepsilon+a_0(a_0-q_0)q_\varepsilon+\\ &\quad c_0b_0a_\varepsilon+c_0a_0b_\varepsilon+b_0a_0c_\varepsilon=0,\\ &\qquad\delta,\gamma,\beta,\varepsilon=12,13,23. \end{aligned}\right\} \quad (63)$$

In symmetric case, when components under 2 generators are equal in all elements $a_{12}=a_{13}=a_{23}$ etc., the system simplifies and comes to the case of 2 elements, as nilpotency index of non-numerical part in both cases equals 2.

As for two Grassmann generators, Eq.(60) will be clarifying, mark 3 cases.

Case I: $d_0=0$, then the system has the form

$$\left.\begin{array}{l} d_0=0,\\ p_0a_0(a_0-p_0)+c_0b_0a_0=0,\\ q_0a_0(a_0-q_0)+c_0b_0a_0=0,\\ c_0a_0d_{12}=0,\ c_0a_0d_{13}=0,\ c_0a_0d_{23}=0,\\ b_0a_0d_{12}=0,\ b_0a_0d_{13}=0,\ b_0a_0d_{23}=0,\\ a_0^2d_{12}=0,\ a_0^2d_{13}=0,\ a_0^2d_{23}=0,\\ p_0^2d_{12}-c_0b_0d_{12}=0,\ p_0^2d_{13}-c_0b_0d_{13}=0,\\ p_0^2d_{23}-c_0b_0d_{23}=0,\ q_0^2d_{12}-c_0b_0d_{12}=0,\\ q_0^2d_{13}-c_0b_0d_{13}=0,\ q_0^2d_{23}-c_0b_0d_{23}=0. \end{array}\right\} \quad (64)$$

and

$$\begin{gathered} p_0a_0(a_\tau-p_\tau)+p_0(a_0-p_0)a_\tau+a_0(a_0-p_0)p_\tau\\ +c_0b_0a_\tau+c_0a_0b_\tau+b_0a_0c_\tau=0,\\ q_0a_0(a_\psi-q_\psi)+q_0(a_0-q_0)a_\psi+a_0(a_0-q_0)q_\psi\\ +c_0b_0a_\psi+c_0a_0b_\psi+b_0a_0c_\psi=0,\\ \tau,\psi=12,13,23. \end{gathered} \quad (65)$$

From Eq.(64) it follows, that $a_0=0$, as we suppose, that ***R***-matrix contains 6 non-zero elements. The system is as follows:

$$\left.\begin{array}{c} d_0=0,\ a_0=0,\\ p_0^2d_{12}-c_0b_0d_{12}=0,\ p_0^2d_{13}-c_0b_0d_{13}=0,\\ p_0^2d_{23}-c_0b_0d_{23}=0,\\ q_0^2d_{12}-c_0b_0d_{12}=0,\ q_0^2d_{13}-c_0b_0d_{13}=0,\\ q_0^2d_{23}-c_0b_0d_{23}=0,\\ p_0^2a_{12}-c_0b_0a_{12}=0,\ p_0^2a_{13}-c_0b_0a_{13}=0,\\ p_0^2a_{23}-c_0b_0a_{23}=0,\\ q_0^2a_{12}-c_0b_0a_{12}=0,\ q_0^2a_{13}-c_0b_0a_{13}=0,\\ q_0^2a_{23}-c_0b_0a_{23}=0. \end{array}\right\} \quad (66)$$

As d_{12}, d_{13}, d_{23} simultaneously cannot be equal to zero (we suppose that all 6 elements of ***R***-matrix do not come to zero, including the complete element $d\neq0$), then Eq.(66) shows that $p_0^2=q_0^2=c_0b_0$. Insert parameters $p_0=q_0=t$, $c_0=r$, then $b_0=t^2/r$, and the components d_{12}, d_{13}, d_{23}, a_{12}, a_{13}, a_{23}, c_{12}, c_{13}, c_{23}, b_{12}, b_{13}, b_{23}, p_{12}, p_{13}, p_{23}, q_{12}, q_{13}, q_{23} remain undefined, i.e. the solutions will be 20-parametrical.

By analogy with expansion on 2 Grassmann generators Case II coincides with Case I, and Case III is obtained from Case I by putting $c_0=b_0=0$.

SOLUTIONS OF YANG-BAXTER EQUATION OVER GRASSMANN ALGEBRA WITH FOUR ELEMENTS

The classification of all solutions was obtained earlier and Cases 1, 2(a), 3(a) require no further study, as the solutions are given in evident form Eqs.(36), (38), (40), (41) for any number of Grassmann algebra generators. The remaining Cases 2(b), 3(b), 3(c), 3(d) contain nilpotent equations. In the general case, for nilpotents it is impossible to give concrete expressions for their solutions. That is why these cases require further analysis.

As expressions of the third degree on Grassmann algebra with four elements equal zero, Eqs.(28)~(31) are simplified in the following way. Case 2(b) contains the equation of the second degree that is why it remains without changes

$$\tilde{p}^2=\tilde{q}^2. \quad (67)$$

For Case 3(b) the system of Eqs.(45)~(46) also remains unchanged, but the expressions are changed for $f=-\tilde{b}/b_0-\tilde{c}/c_0$.

In Case 3(c), Eq.(48) is quadratic, which is why they do not change

$$ab=0,\ db=0,\ ad=0. \quad (68)$$

In Case 3(d) all parameters p, q, a, b, c, d can be arbitrary, even nilpotent.

For the solutions of these systems, let us write all the parameters in the component form

$$x=\sum_{\substack{i,j=1\\ i<j}}^{4}x_{ij}\xi_i\xi_j+x_{1234}\xi_1\xi_2\xi_3\xi_4, \quad (69)$$

where x=a, b, c, d, p, q. Coordinate x_{1234} can be arbitrary, as it cancels any such nilpotent. Then all the components in the equation will be as follows

Case 2(b)

$$p_{12}p_{34}+p_{14}p_{23}-p_{13}p_{24}=q_{12}q_{34}+q_{14}q_{23}-q_{13}q_{24}. \quad (70)$$

The solution can be, for example, under $p_{12}\neq 0$

$$p_{34}=(-p_{14}p_{23}+p_{13}p_{24}+q_{12}q_{34}+q_{14}q_{23}-q_{13}q_{24})/p_{12}. \quad (71)$$

Case 3(b)

$$\begin{aligned}
&q_{12}a_{34}-q_{13}a_{24}+q_{14}a_{23}+q_{23}a_{14}-q_{24}a_{13}+q_{34}a_{12}=0,\\
&q_{12}d_{34}-q_{13}d_{24}+q_{14}d_{23}+q_{23}d_{14}-q_{24}d_{13}+q_{34}d_{12}=0,\\
&[2(a_{12}q'_{13}-a_{13}q'_{12})+(q_{12}f_{13}-q_{13}f_{12})]a_{24}-\\
&[2(a_{12}q'_{14}-a_{14}q'_{12})+(q_{12}f_{14}-q_{14}f_{12})]a_{23}-\\
&[2a_{12}q'_{23}+(q_{12}f_{23}-q_{23}f_{12})]a_{14}+\\
&[2a_{12}q'_{24}+(q_{12}f_{24}-q_{24}f_{12})]a_{13}-\\
&[2a_{12}q'_{34}+(q_{12}f_{34}-q_{34}f_{12})]a_{12}=0,\\
&[2(d_{12}q'_{13}-d_{13}q'_{12})+(q_{12}f_{13}-q_{13}f_{12})]d_{24}-\\
&[2(d_{12}q'_{14}-d_{14}q'_{12})+(q_{12}f_{14}-q_{14}f_{12})]d_{23}-\\
&[2d_{12}q'_{23}+(q_{12}f_{23}-q_{23}f_{12})]d_{14}+\\
&[2d_{12}q'_{24}+(q_{12}f_{24}-q_{24}f_{12})]d_{13}-\\
&[2d_{12}q'_{34}+(q_{12}f_{34}-q_{34}f_{12})]d_{12}=0,\\
&(a_{12}q'_{13}-a_{13}q'_{12})d_{24}-(a_{12}q'_{14}-a_{14}q'_{12})d_{23}-\\
&(a_{12}q'_{23}-a_{23}q'_{12})d_{14}+(a_{12}q'_{24}-a_{24}q'_{12})d_{13}+\\
&[-2a_{12}q'_{34}+(a_{13}q'_{24}+a_{24}q'_{13})-(a_{14}q'_{23}+a_{23}q'_{14})]d_{12}=0,
\end{aligned}$$

here $q'=\tilde{q}$ in the case "+" and $q'=\tilde{q}-f$ in the case "–".

Case 3(c)

$$\begin{aligned}
&-b_{12}a_{34}+b_{13}a_{24}-b_{14}a_{23}-b_{23}a_{14}+b_{24}a_{13}-b_{34}a_{12}=0,\\
&-b_{12}d_{34}+b_{13}d_{24}-b_{14}d_{23}-b_{23}d_{14}+b_{24}d_{13}-b_{34}d_{12}=0,\\
&(a_{12}b_{13}-a_{13}b_{12})d_{24}-(a_{12}b_{14}-a_{14}b_{12})d_{23}-\\
&(a_{12}b_{23}-a_{23}b_{12})d_{14}+(a_{12}b_{24}-a_{24}b_{12})d_{13}+\\
&[-2a_{12}b_{34}+(a_{13}b_{24}+a_{24}b_{13})-(a_{14}b_{23}+a_{23}b_{14})]d_{12}=0.
\end{aligned}$$

This system of equations can be solved, for example, if $b_{12}\neq 0$ and $(a_{12}b_{13}-a_{13}b_{12})\neq 0$. Then we can express a_{34}, d_{34}, d_{24} through other variables. The obtained systems for such coordinates are linear, that is why their solutions are rational functions of the rest of the components. Final expressions are too cumbersome to be presented here.

Only in Cases 3(b), 3(c), 3(d) there exist 6-vertex solutions, in contrast to the solutions over the numerical field in which only 5-vertex solutions are possible (Hietarinta, 1992; 1993).

REGULAR YANG-BAXTER EQUATION SOLUTIONS

In (Duplij and Sadovnikov, 2002) set-theoretical (Drinfeld, 1992; Etingof *et al*., 1999; Gu, 1997) regular (according to von Neumann) solutions for Yang-Baxter equation were found. The consideration of similar solutions for 6-vertex $\boldsymbol{R}$-matrix is interesting. Let us remind that matrix $\boldsymbol{R}$ is regular (von Neumann), if there exists matrix $\bar{\boldsymbol{R}}$ such that $\boldsymbol{R}\bar{\boldsymbol{R}}\boldsymbol{R}=\boldsymbol{R}, \bar{\boldsymbol{R}}\boldsymbol{R}\bar{\boldsymbol{R}}=\bar{\boldsymbol{R}}$ (Penrose, 1955; Rao and Mitra, 1971; Clifford and Preston, 1972). Such $\boldsymbol{R}$-matrices appeared during the study of weak Hopf algebras (Li and Duplij, 2002; Duplij and Li, 2001a; 2001b). Properties of regular supermatrices were considered in (Duplij, 2000; Duplij and Kotulskaja, 2002).

For 6-vertex $\boldsymbol{R}$-matrix we will use, instead of the "reversibility" condition

$$\boldsymbol{R}^{21}\boldsymbol{R}=\boldsymbol{R}\,\boldsymbol{R}^{21}=id, \quad (72)$$

which is called the unitarity in (Etingof *et al*., 1997; 1999), the Von Neumann regularity in the form

$$\boldsymbol{R}\bar{\boldsymbol{R}}^{21}\boldsymbol{R}=\boldsymbol{R},\ \bar{\boldsymbol{R}}^{21}\boldsymbol{R}\bar{\boldsymbol{R}}^{21}=\bar{\boldsymbol{R}}^{21}, \quad (73)$$

where

$$\bar{\boldsymbol{R}}^{21}=\begin{pmatrix} p & \cdot & \cdot & \cdot \\ \cdot & b & a & \cdot \\ \cdot & d & c & \cdot \\ \cdot & \cdot & \cdot & q \end{pmatrix}. \quad (74)$$

Notice that non-unitary [with violation Eq.(72)] set-theoretical solutions to Yang-Baxter equation of another form were considered in (Lu *et al*., 2000; Soloviev, 2002).

Substitution of Eq.(74) into Eq.(73) gives limitations, on the condition of regularity on the elements of 6-vertex $\boldsymbol{R}$-matrix Eq.(26)

$$
\begin{aligned}
&p^3 = p, q^3 = q,\\
&(ab+bd)d + (a^2+cb)b = b,\\
&(ab+bd)c + (a^2+cb)a = a,\\
&(cb+d^2)d + (ca+dc)b = d,\\
&(cb+d^2)c + (ca+dc)a = c.
\end{aligned} \tag{75}
$$

The analysis shows that regular noninvertible solutions (satisfying $\boldsymbol{R}^{21}\boldsymbol{R}\neq id$) can appear in cases: (1) under a=±1, p and q are simultaneously not equal to a; 2(a) under d=±1, a=0, bc=0, p={0, ±1}, q={0, ±1}. For example, for a particular case from Case 1 we have

$$
\boldsymbol{R} = \begin{pmatrix} a & 0 & 0 & 0\\ 0 & 0 & a & 0\\ 0 & a & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{R}}^{21} = \begin{pmatrix} a & 0 & 0 & 0\\ 0 & 0 & a & 0\\ 0 & a & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\boldsymbol{R}\bar{\boldsymbol{R}}^{21} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & a^2 & 0 & 0\\ 0 & 0 & a^2 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \neq id. \tag{76}
$$

but regularity Eq.(73) is fulfilled. In a particular case from Case 2(a) we get

$$
\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & c & 1 & 0\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{\boldsymbol{R}}^{21} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 1 & c & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\boldsymbol{R}\bar{\boldsymbol{R}}^{21} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & c & 0\\ 0 & b & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \neq id. \tag{77}
$$

and regularity Eq.(73) is fulfilled.

Thus, we presented the classification of constant 6-vertex Yang-Baxter equation solutions. Concrete examples were given for Cases 2, 3, 4 of Grassmann algebra elements and the corresponding analysis was done. The obtained solutions can be used for supersymmetric quantum group generalizations and quantum computations.

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